2013-2014 MM2MS2 Exam Solutions

1.

(a)

$F = 60{,}000\ N$

$T = 3500\ Nm$

$d_o = 61.5\ mm = 0.0615\ m$

$d_i = 55.5\ mm = 0.0555\ m.$

$\left(\begin{array}{l} r_o = 0.03075\ m \\ r_i = 0.02775\ m \end{array} \right)$

Axial force, $\sigma_{ax} = \frac{F}{A}$

$$= \frac{F}{\pi (r_o^2 - r_i^2)}$$

$$= \frac{60000}{\pi (0.03075^2 - 0.02775^2)}$$

$$= 1.09 \times 10^8\ Pa$$

$$= \underline{\underline{109\ MPa}}.$$ [2 marks].

Shear stress $\tau = \frac{Tr}{J}$

$$J = \frac{\pi (0.0615^4 - 0.0555^4)}{32}$$

$$J = 4.72 \times 10^{-7}\ m^4.$$

$$\tau = \frac{3500 \times 0.03075}{4.72 \times 10^{-7}}$$

$$= 2.27 \times 10^8\ Pa$$

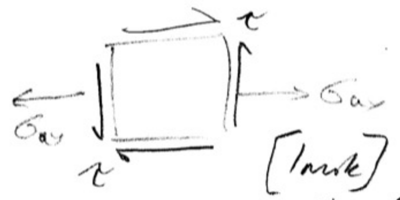
$$= \underline{\underline{227\ MPa}}$$ [2 marks].

(b)

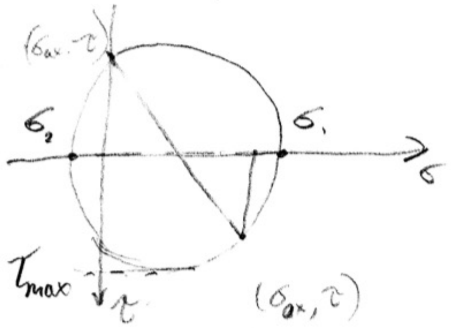
Stresses at A

$\sigma_{ax} = 109\,MPa$.
$\tau = 227\,MPa$.
$\sigma_h = 0$.

Mohr's Circle



[1 mark]
Stress State.



Drg. Mohrs Circle [1 mark]

$$C = \left(\frac{\sigma_{ax} + \sigma_h}{2}\right)$$

$$C = \frac{\sigma_{ax}}{2}$$

$$C = \frac{109}{2}$$

$$C = 54.5\,MPa.$$

$\sigma_1 = C + R$

$\sigma_2 = C - R$

$\tau_{max} = R$.

$$R = \sqrt{\left(\frac{\sigma_{ax}}{2}\right)^2 + \tau^2}$$

$$= \underline{233\,MPa = \tau_{max}}$$ [2 marks].

$\sigma_1 = C + R = \underline{287\,MPa}$. [1 mark]

$\sigma_2 = C - R = \underline{-178\,MPa}$ [1 mark].

(c)

$$\sigma_{ax} = \frac{F}{A} = \frac{100000}{\pi(0.03075^2 - 0.02775^2)}$$

$$\sigma_{ax} = 1.81 \times 10^8 \text{ Pa}$$

$$= \underline{\underline{181 \text{ MPa}}}$$

[1 mark]

$$\tau = \frac{Tr}{J} = \frac{5000 \times 0.03075}{4.72 \times 10^{-7}}$$

$$= 3.26 \times 10^8 \text{ Pa}$$

$$= \underline{\underline{326 \text{ MPa}}}$$

[1 Mark]

$$\sigma_B = \pm \frac{My}{I}$$ ← distance from N.A. = radius.

$$I = \frac{\pi(d_o^4 - d_i^4)}{64}$$

$$= \frac{\pi \times (0.0615^4 - 0.0555^4)}{64}$$

$$= 2.36 \times 10^{-7} \text{ m}^4$$

$$\sigma_B = \pm \frac{1500 \times 0.03075}{2.36 \times 10^{-7}}$$

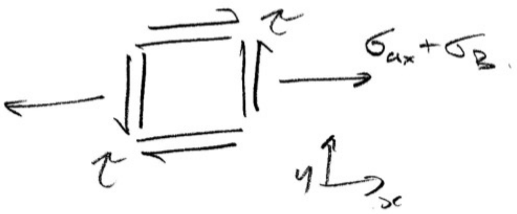
$$= \pm 1.95 \times 10^8 \text{ Pa}$$

$$= \underline{\underline{\pm 195 \text{ MPa}}}$$

Point A −195 MPa. (compressive)
Point B +195 MPa (tensile).

[3 marks].

Point A



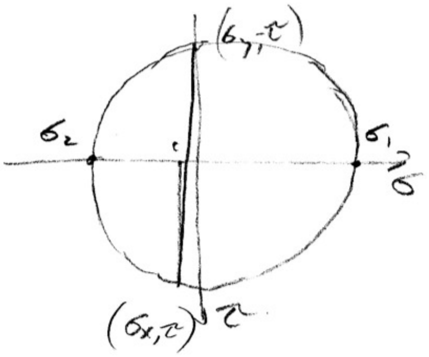
Assessing Stress state. [1 mark].

$\sigma_x = \sigma_{ax} + \sigma_B$
$= 181 - 195$
$= -14\,MPa$

$\tau = 326\,MPa$

$\sigma_y = 0$

Mohr's Circle



Drg. Mohr's circle. [1 mark].

$C = \frac{\sigma_x}{2}$

$= \frac{14}{2} = 7\,MPa.$

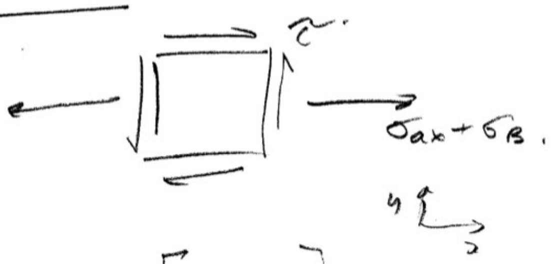
$R = \sqrt{7^2 + 326^2}$

$R = 326\,MPa = \tau_{max}$ [1 mark]

$\sigma_1 = C + R = 7 + 326 = 333\,MPa$ [1 mark]

$\sigma_2 = C - R = 7 - 326 = -319\,MPa$ [1 mark]

Point B.



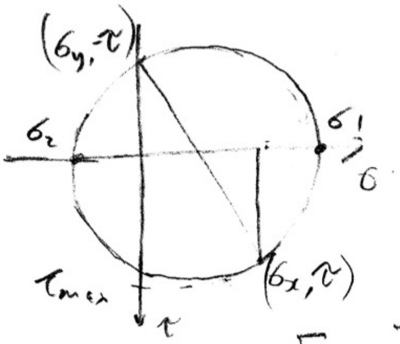
Stress state. [1 mark].

$\sigma_x = \sigma_{ax} + \sigma_B.$
$= 181 + 195$
$= 376\,MPa$

$\tau = 326\,MPa$

$\sigma_y = 0$

Mohr's Circle



Drg Mohrs Circle [1 mark]

$$C = \frac{\sigma_x}{2} = 188 \text{ MPa}$$

$$R = \sqrt{188^2 + 326^2}$$

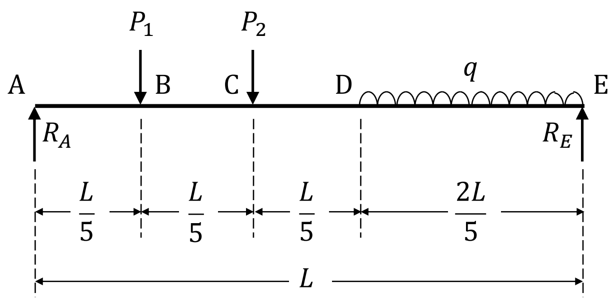
$$R = 376 \text{ MPa} = \tau_{max} \quad [1 \text{ mark}]$$

$$\sigma_1 = C + R = 188 + 376 = 564 \text{ MPa} \quad [1 \text{ mark}]$$

$$\sigma_2 = C - R = 188 - 376 = -187 \text{ MPa}. \quad [1 \text{ mark}]$$

2.

(a)



[1 mark]

Vertical equilibrium of the beam:

$$R_A + R_E = P_1 + P_2 + \frac{2qL}{5} \qquad (1)$$

[1 mark]

Taking moments about position E:

$$\frac{4P_1L}{5} + \frac{3P_2L}{5} + \frac{4qL^2}{50} = R_AL$$

[1 mark]

Substituting values of P_1, P_2, q and L gives:

$$\boldsymbol{R_A = 10,600\ \mathrm{N}}$$

[1 mark]

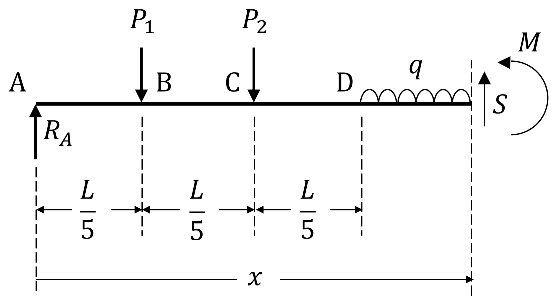
Rearranging (1) for R_E and substituting values for R_A, P_1, P_2, q and L gives:

$$\boldsymbol{R_E = 16,400\ \mathrm{N}}$$

[1 mark]

(b)

Taking the origin at the left-hand end of the beam, sectioning after the last discontinuity and drawing a free body diagram of the left-hand side of the section:



[3 marks]

(c)

Taking moments about the section position:

$$M + P_1 \langle x - \frac{L}{5} \rangle + P_2 \langle x - \frac{2L}{5} \rangle + \frac{q \langle x - \frac{3L}{5} \rangle^2}{2} = R_A x$$

$$\therefore M = R_A x - P_1 \langle x - \frac{L}{5} \rangle - P_2 \langle x - \frac{2L}{5} \rangle - \frac{q \langle x - \frac{3L}{5} \rangle^2}{2}$$

[1 mark]

Substituting this into the main deflections of beams equation ($EI \frac{d^2y}{dx^2} = M$):

$$EI \frac{d^2y}{dx^2} = R_A x - P_1 \langle x - \frac{L}{5} \rangle - P_2 \langle x - \frac{2L}{5} \rangle - \frac{q \langle x - \frac{3L}{5} \rangle^2}{2}$$

[1 mark]

Integrating with respect to x:

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - \frac{P_1 \langle x - \frac{L}{5} \rangle^2}{2} - \frac{P_2 \langle x - \frac{2L}{5} \rangle^2}{2} - \frac{q \langle x - \frac{3L}{5} \rangle^3}{6} + A \qquad (2)$$

[1 mark]

Integrating with respect to x again:

$$EIy = \frac{R_A x^3}{6} - \frac{P_1 \langle x - \frac{L}{5} \rangle^3}{6} - \frac{P_2 \langle x - \frac{2L}{5} \rangle^3}{6} - \frac{q \langle x - \frac{3L}{5} \rangle^4}{24} + Ax + B \qquad (3)$$

[1 mark]

Boundary conditions:

(BC1) At $x = 0$, $y = 0$, therefore from (3):

$$B = 0$$

[1 mark]

(BC2) At $x = L$, $y = 0$, therefore from (3):

$$0 = \frac{R_A L^3}{6} - \frac{64P_1 L^3}{750} - \frac{27P_2 L^3}{750} - \frac{16qL^4}{15000} + AL$$

$$\therefore A = \frac{64P_1 L^3}{750} + \frac{27P_2 L^3}{750} + \frac{16qL^4}{15000} - \frac{R_A L^3}{6}$$

Substituting values of P_1, P_2, L, q, and R_A into this gives:

$$A = -28{,}266.67$$

[1 mark]

From (3), at $x = \frac{2L}{5}$ (point C):

$$y = \frac{1}{EI}\left(\frac{8R_A L^3}{750} - \frac{P_1 L^3}{750} + \frac{2AL}{5}\right)$$

[1 mark]

Substituting values of E, I, R_A, L, P_1 and A into this gives:

$$\boldsymbol{y = -0.01025\ \mathrm{m} = -10.25\ \mathrm{mm}}$$

(i.e. downward deflection)

[2 marks]

From (2), at $x = \frac{2L}{5}$ (point C):

$$\frac{dy}{dx} = \frac{1}{EI}\left(\frac{4R_A L^2}{50} - \frac{P_1 L^2}{50} + A\right)$$

[1 mark]

Substituting values of E, I, R_A, L, P_1 and A into this gives:

$$\boldsymbol{\frac{dy}{dx} = -0.00216\ \mathrm{rad} = -0.12\,°}$$

(i.e. small negative gradient)

[2 marks]

(d)

From (3), at $x = \frac{4L}{5}$:

$$y = \frac{1}{EI}\left(\frac{64R_A L^3}{750} - \frac{27P_1 L^3}{750} - \frac{8P_2 L^3}{750} - \frac{qL^4}{15000} + \frac{4AL}{5}\right)$$

[2 marks]

Substituting values of E, I, R_A, L, P_1, P_2, q and A into this gives:

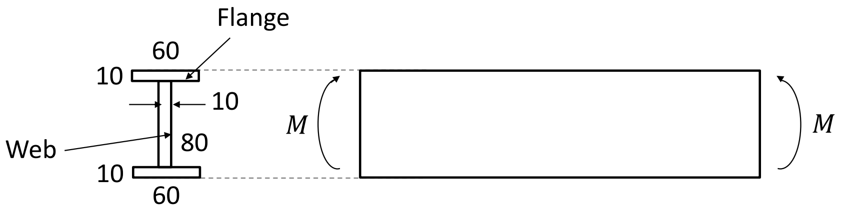
$$\boldsymbol{y = -0.00658\ \mathrm{m} = -6.58\ \mathrm{mm}}$$

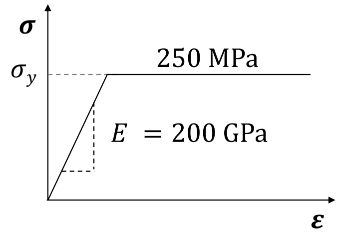
(i.e. downward deflection)

[3 marks]

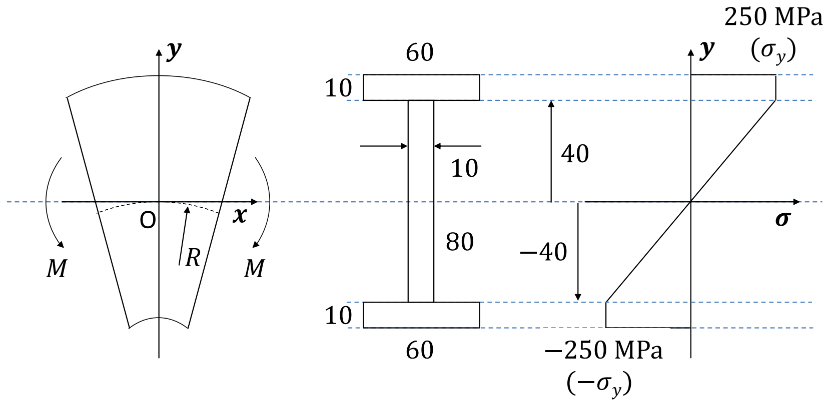
3.

(a)





Yielding will occur through whole of flange, therefore:



Section of the beam in bending

Cross section of the beam

Stress distribution through the beam cross section

Variation of stress with y:

- For a $< y < 50, \sigma = 250$ MPa
- For $-$a $< y < a, \sigma = \frac{250}{40} y$ MPa
- For $-50 < y < -a, \sigma = -250$ MPa

[3 marks]

Moment equilibrium:

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment)

$$M = \int_A y\sigma dA = \int_y y\sigma b dy$$

[2 mark]

Due to the symmetry of the stress distribution and substituting in the elastic and plastic terms for σ, this can be rewritten as:

$$M = 2\left\{\int_0^{40} y\frac{250}{40}y(10)dy + \int_{40}^{50} y(250)(60)dy\right\} = 2\left\{\frac{250}{4}\int_0^{40} y^2dy + 15{,}000\int_{40}^{50} ydy\right\}$$

$$= 2\left\{\frac{250}{4}\left[\frac{y^3}{3}\right]_0^{40} + 15{,}000\left[\frac{y^2}{2}\right]_{40}^{50}\right\} = 2\left\{\frac{250}{4}\left(\frac{40^3}{3}\right) + 15{,}000\left(\frac{50^2}{2} - \frac{40^2}{2}\right)\right\}$$

$$\therefore \boldsymbol{M = 16{,}166{,}666.66\ \mathbf{Nmm} = 16.17\ \mathbf{kNm}}$$

[3 marks]

<u>Compatibility</u>

$$\varepsilon = \frac{y}{R} \qquad (1)$$

[1 mark]

At $y = 40$ mm, $\sigma = \sigma_y = 200$ MPa and since this point is within the elastic range:

$$\varepsilon = \frac{\sigma_y}{E} = \frac{250}{200{,}000} = 1.25 \times 10^{-3}$$

[1 mark]

Substituting this into (1) gives:

$$1.25 \times 10^{-3} = \frac{40}{R}$$

$$\therefore \boldsymbol{R = 32{,}000\ \mathbf{mm} = 32\ \mathbf{m}}$$

[3 marks]

(b)

$$I = \left(\frac{bd^3}{12}\right)_{outer} - \left(\frac{bd^3}{12}\right)_{gaps} = \frac{60 \times 100^3}{12} - 2\left(\frac{25 \times 80^3}{12}\right) = 2{,}866{,}666.67\ \mathrm{mm}^4$$

[2 marks]

Unloading is assumed to be entirely elastic. Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta\sigma}{y}$$

[1 mark]

Max change in stress ($\Delta\sigma$) will occur at $y\ = \frac{d}{2} = y_{max}$ ($= \pm 50$ mm).

$$\therefore\ \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-16{,}166{,}666.66 \times \pm 50}{2{,}866{,}666.67}$$

$$= \mp 281.98\ \text{MPa}$$

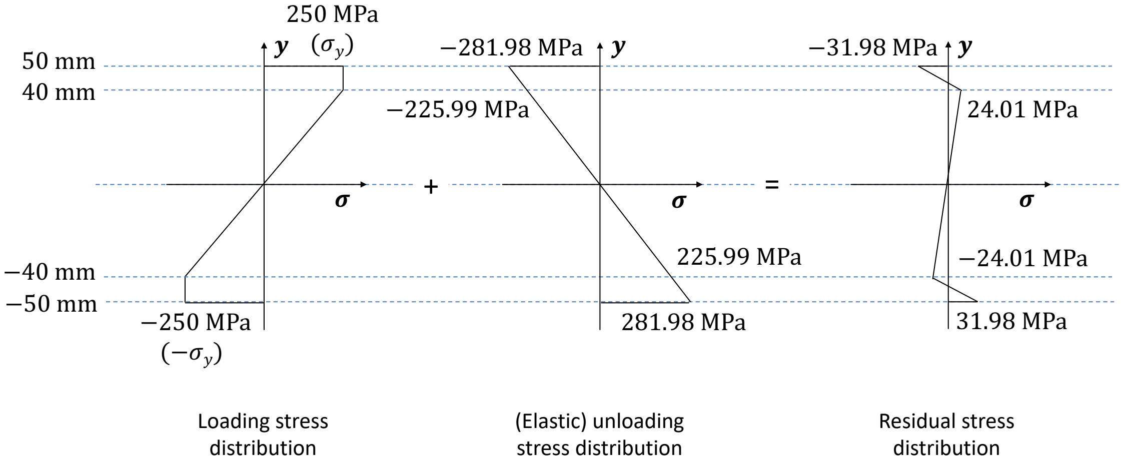
i.e. at $y = 50$ mm

$$\Delta\sigma_{max}^{el} = -281.98\ \text{MPa}$$

and at $y = -50$ mm

$$\Delta\sigma_{max}^{el} = 281.98\ \text{MPa}$$

[2 marks]



Interpolation of (elastic) unloading line:
At $y = 50$ mm, $\sigma = -281.98$ MPa
$y = m\sigma + c$
$\therefore 50 = \text{m} \times -281.98 + 0$
$\therefore \text{m} = -0.177$
At $y = 40$ mm, $40 = -0.417 \times \sigma$
$\therefore \sigma = -225.99$ MPa

[2 marks]

Residual stress is well below yield (250 MPa), so reverse yielding does not occur. At $y = 40$ mm, no plastic deformation occurs during loading and unloading,

$$\varepsilon_{residual} = \frac{\sigma_{residual}}{E} = \frac{24.01}{200{,}000} = 1.201 \times 10^{-4}$$

[1 mark]

Also,

$$\varepsilon = \frac{y}{R}$$

[1 mark]

$$\therefore 1.201 \times 10^{-4} = \frac{40}{R}$$

$$\therefore \boldsymbol{R = 333{,}055.798\ \mathrm{mm} = 333.06\ \mathrm{m}}$$

[3 marks]

4.

(a)

Stress Intensity Factor is given as:

$$K_I = Y\sigma\sqrt{\pi a}$$

where the geometry (and therefore boundaries) affect the value of Y. For example, for a for a crack in an infinite plate, $Y = 1$ and for small values of a/W, $Y = 1.12$ (where W is the width of the plate).

[3 marks]

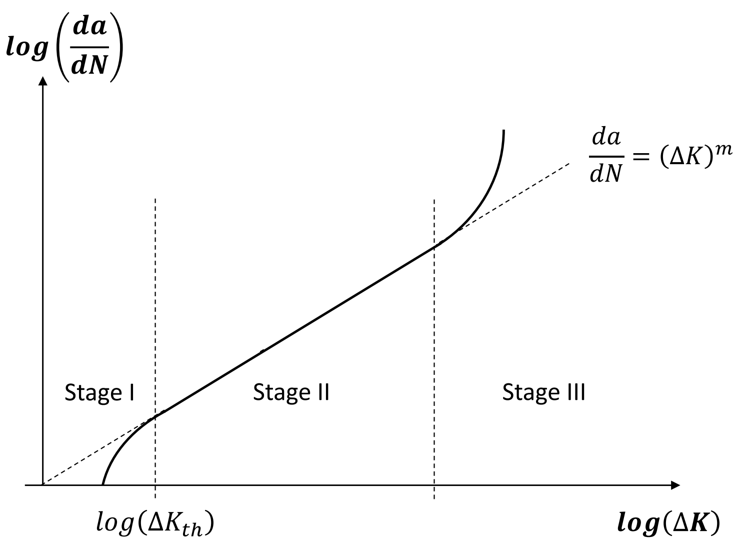
(b)

Paris showed that crack growth can be represented by the following empirical relationship:

$$\frac{da}{dN} = C(\Delta K)^m$$

[2 marks]

Where C and m are empirically determined material constants. There are 3 stages in the relationship between Crack growth Rate and Stress Intensity Factor, as shown in the following figure.



[3 marks]

Stage I: Below ΔK_{th}, no observable crack growth occurs.

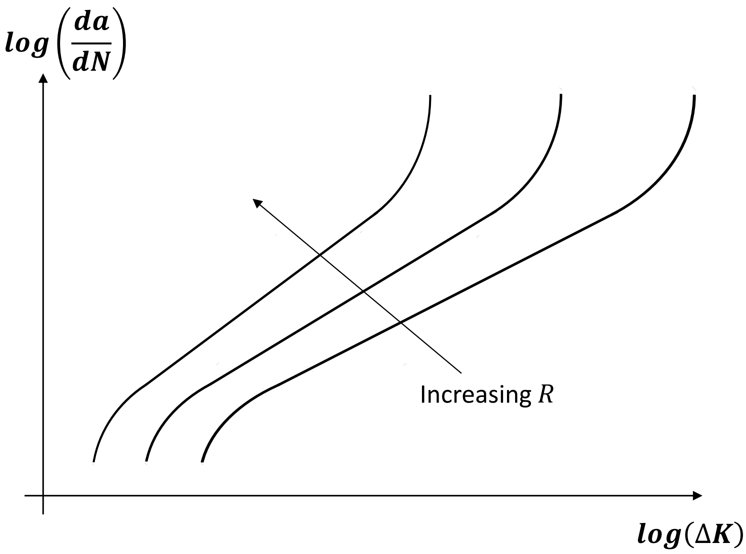
Stage II: This region shows an essentially linear relationship between Crack growth Rate and Stress Intensity Factor (on a log-log scale), where m is the slope and C is the vertical axis intercept.

Stage III: Rapid crack growth occurs, and little life is involved.

[1 mark]

(c)

As mean stress (and therefore R) is increased, fatigue life is decreased as shown in the following figure:



[6 marks]

(d)

Failure when $K > K_C$

[2 marks]

Therefore, $K < {}^{K_C}/_2$ for inclusion of safety factor. Where,

$$K = Y\sigma\sqrt{\pi a}$$

[2 marks]

Rearranging,

$$\sigma = \frac{{}^{K_C}/_2}{Y\sqrt{\pi a}} = \frac{{}^{75}/_2}{1.12\sqrt{\pi \times 8 \times 10^{-3}}} = 211.2 \text{ MPa}$$

[1 mark]

This component is a thin walled cylinder, therefore,

$$\sigma_\theta = \frac{PR}{t}$$

[2 marks]

Rearranging,

$$P = \frac{\sigma_\theta t}{R} = \frac{211.2 \times 10}{1000/_2} = \mathbf{4.22\ MPa = 42.2\ bar}$$

[3 marks]

5.

(a)

$$\varepsilon_{thermal} = \frac{l\alpha T}{l} = \alpha T$$

(b)

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) + \alpha T$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

etc...

(c)

Axial force equilibrium,

$$P = E\bar{\varepsilon}A - E\alpha \int_A T dA$$

$$0 = E\bar{\varepsilon}bd - E\alpha \int_{-d/2}^{d/2} T_o\left(1 - \frac{4y^2}{d^2}\right) bdy$$

$$\therefore \bar{\varepsilon} = \frac{\alpha}{d}T_o \int_{-d/2}^{d/2}\left(1 - \frac{4y^2}{d^2}\right)dy = \frac{\alpha}{d}T_o\left[y - \frac{4y^3}{3d^2}\right]_{-d/2}^{d/2} = \frac{2}{3}\alpha T_o$$

With $M = 0$ we can obtain $1/R$ from the moment equilibrium but from symmetry we can see that $1/R = 0$.

$$\sigma_x = E\left(\bar{\varepsilon} + \frac{y}{R} - \alpha T\right) = E\left(\frac{2}{3}\alpha T_o - \alpha T_o\left(1 - \frac{4y^2}{d^2}\right)\right) = E\alpha T_o\left(\frac{4y^2}{d^2} - \frac{1}{3}\right)$$

At $y = 0$,

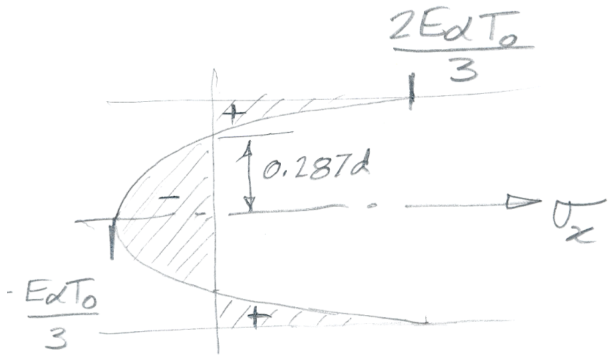
$$\sigma_x = -\frac{E\alpha T_o}{3}$$

At $y = \pm\frac{d}{2}$,

$$\sigma_x = \frac{2E\alpha T_o}{3}$$

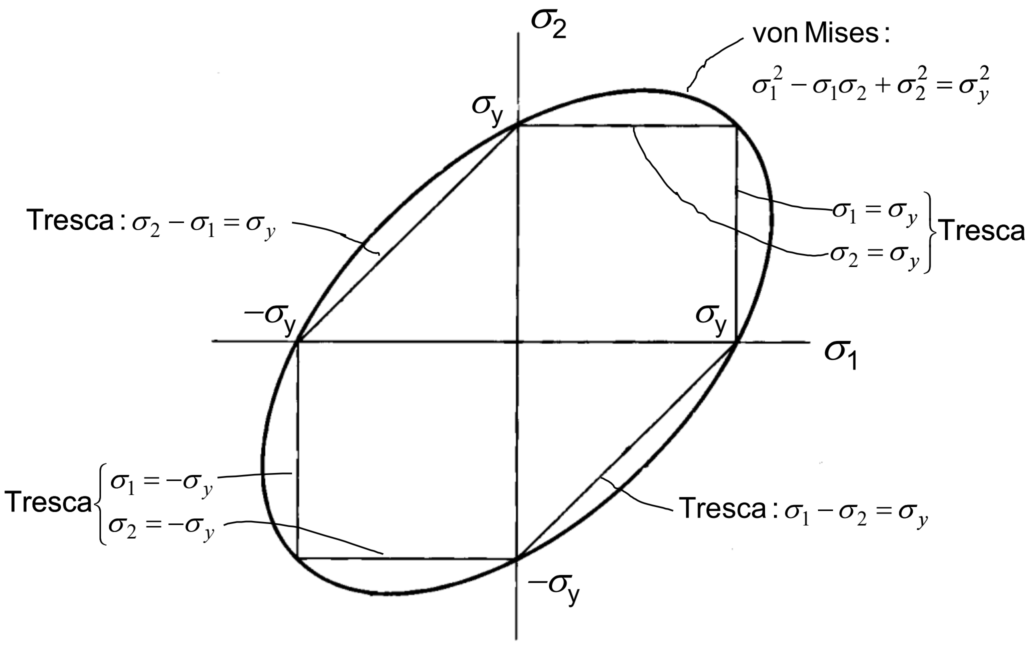
$\sigma_x = 0$ when $\frac{4y^2}{d^2} = \frac{1}{3}$, i.e. at $y = \pm 0.287d$

This is the stress distribution away from the ends. At the ends, $\sigma_x = 0$ and there is a gradual transition between these:

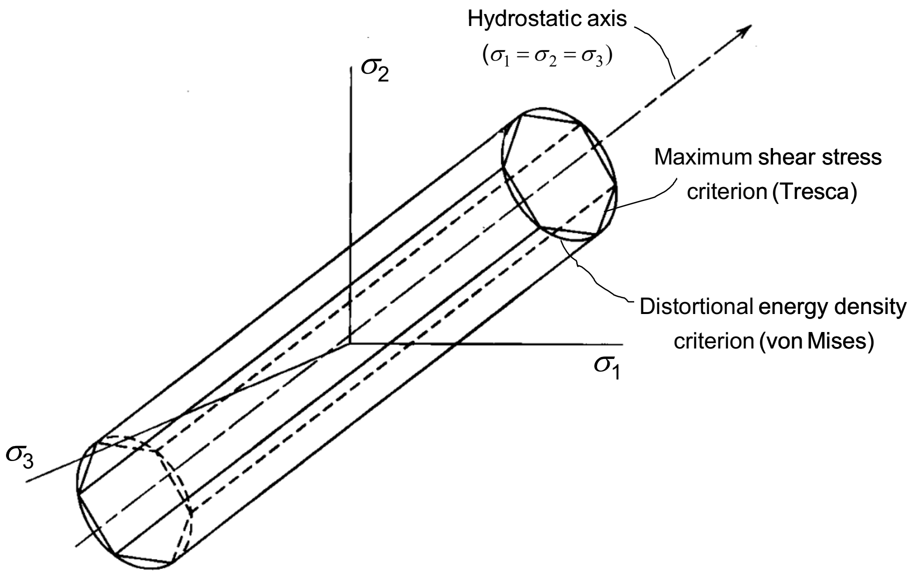


6.

(a)



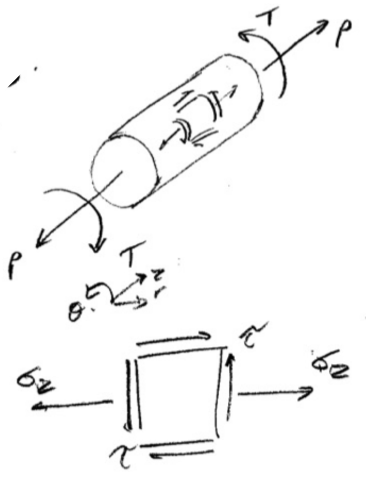
(b)



(c)

See part (b).

(d)

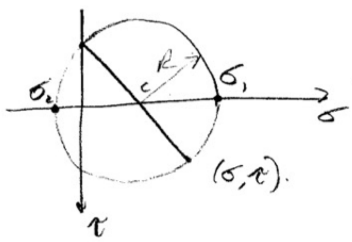


Yield Stress (σ_y) = 250 MPa

$$\sigma_z = \frac{F}{A} = \frac{195 \times 10^3 \, N}{\pi \times 25 \, mm^2}$$

$$\sigma_z = \underline{99.3 \, MPa}.$$

Mohr's Circle:



$\sigma_z = 99.3 \, MPa$

$\sigma_\theta = 0$.

$\tau = ?$

σ_1 is +ve

σ_2 is -ve as $\sigma_\theta = 0$

$\therefore$ σ_3 will be intermediate as σ_2 is minimum principal stress so, $\sigma_1 > \sigma_3 > \sigma_2$ in this case.

so, for Tresca yield criterion

$$\sigma_1 - \sigma_2 = \sigma_y \qquad (1)$$

from Mohr's Circle $\quad C = \frac{\sigma_z}{2} = \frac{99.3}{2} = 49.65 \, MPa.$

$$R = \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau^2}$$

$$\sigma_1 = C + R \quad (2)$$
$$\sigma_2 = C - R \quad (3).$$

∴ subs. (2) & (3) into (1).

$$\sigma_y = (C+R) - (C-R)$$

$$\Rightarrow \sigma_y = 2R \quad \text{at yield.}$$

$$\sigma_y = 2\sqrt{49.65^2 + \tau^2}$$

$$\left(\frac{\sigma_y}{2}\right)^2 = 49.65^2 + \tau^2$$

$$\Rightarrow \tau = \sqrt{\left(\frac{\sigma_y}{2}\right)^2 - 49.65^2}$$

$$\tau = \underline{\underline{114.7\,MPa}}. \quad \boxed{\text{Tresca}}$$

for elastic range

$$\tau = \frac{Tr}{J} \Rightarrow T = \frac{\tau J}{r} \quad (4)$$

$$J = \frac{\pi d^4}{32}$$
$$= \frac{\pi \times 50^4}{32}$$
$$= 6.14 \times 10^5\ mm^4$$

$$\Rightarrow T = \frac{114.7 \times 6.14 \times 10^5}{25} = \underline{\underline{2.82 \times 10^6\ Nmm}}$$

$$\boxed{\text{Tresca}}$$

$$= \underline{\underline{2.8\,kNm}}$$

for **von Mises** yield criterion

$\sigma_2 = 0 \quad \therefore \quad \sigma_3 = 0$

$(\sigma_1 + \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$ at yield.

reduces to:

$(\sigma_1 - \sigma_2)^2 + \sigma_2^2 + \sigma_1^2 = 2\sigma_y^2$

$\Rightarrow \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2 \quad (5)$

subs (2) & (3) into (5)

$(C+R)^2 + (C-R)^2 - (C+R)(C-R) = \sigma_y^2$

$C^2 + R^2 + 2CR + C^2 + R^2 - 2CR - C^2 + R^2 = \sigma_y^2.$

$\Rightarrow \sigma_y^2 = C^2 + 3R^2$

$\Rightarrow R = 141.46$

$R = \sqrt{\left(\frac{\sigma_z}{2}\right)^2 + \tau^2}$

$\Rightarrow \tau = \underline{\underline{132.5 \text{ MPa}}}$ **von Mises**

$T = \frac{132.5 \times 6.14 \times 10^5}{25} = \underline{\underline{3.25 \times 10^6 \text{ Nmm}}}$

$= \underline{\underline{3.25 \text{ kNm}}}$ **von Mises**